Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 213 (2008) 408–422
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Algorithms for infinite quadratic programming in L_p spaces

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Received 27 October 2006; received in revised form 11 January 2007

Abstract

We study infinite dimensional quadratic programming problems of an integral type. The decision variable is taken in the L_p space where $1 < p < \infty$. In this paper the decision variable is required to have a lower bound and an upper bound on a compact interval. Two numerical algorithms are proposed for solving these problems, and the convergence properties of the proposed algorithms are given. Two numerical examples are also given to implement the proposed algorithms.

MSC: 90C20; 49M20; 49M25

Keywords: Infinite quadratic programming in L_p spaces; Cutting plane method; Algorithms; Optimal value; Optimal solution; Discretization

1. Introduction

Let X and Y be compact intervals. For $p \geqslant 1$, the space $L_p(X)$ consists of those real-valued measurable functions f on the compact interval X for which $|f(x)|^p$ is a Lebesgue integrable function. The norm on this space is defined as $\|f\|_{L_p} = (\int_X |f(x)|^p \, dx)^{1/p}$, and we call $\|f\|_{L_p}$ the L_p-norm of f. Now we consider the following infinite dimensional quadratic programming problem. Let $\phi(s, y)$ be a real-valued continuous function on $X \times Y$, $g(y)$ be a real-valued continuous function on Y, $h(s)$ be a real-valued continuous function on X, and $f(s, t)$ be a real-valued continuous function on $X \times X$. Then the infinite dimensional quadratic programming problem (P) is as follows:

$$\begin{aligned} \min_{k \in L_p(X)} \quad & \frac{1}{2} \int_X \int_X f(s,t) k(s) \, ds \, k(t) \, dt + \int_X h(s) k(s) \, ds \\ \text{s.t.} \quad & \int_X \phi(s, y) k(s) \, ds \geqslant g(y) \quad \text{for each } y \in Y, \\ & 0 \leqslant M_1 \leqslant k(s) \leqslant M_2 \text{ a.e.} \quad \text{on } X. \end{aligned}$$

Here, M_1 and M_2 are given constants. In this paper, we only consider the case that $1 < p < \infty$. This is an infinite dimensional quadratic programming problem of an integral type. Lai and Wu [6] studied the infinite dimensional linear programming problems on measure spaces, and the necessary and sufficient conditions for a measure to be optimal

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0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.01.018

were established in their paper. Meanwhile, solving the general capacity problem by relaxed cutting plane approach can be found in Fang et al. [3]. Ito et al. [5] considered infinite dimensional linear programs in L_1 spaces, while Vanderbei [11] investigated an optimization problem for the best high-contrast apodization. This is an infinite dimensional linear programming problem in which the decision variable has a lower bound and an upper bound. Infinite dimensional quadratic programming programs on measure spaces were proposed in Wu [12]. In that paper Wu provided a cutting plane approach to solving quadratic infinite programs on measure spaces. In this paper, we study infinite dimensional quadratic programming problems in the L_p space where $1<p<\infty$, and we require that the decision variable in the L_p space where $1<p<\infty$ has a lower bound and an upper bound on a compact interval. These types of problems are related to Vanderbei's study. Here, we also review [1,2,4,7,10] for our research of this paper.

In the following, $L_q(X)$, $1<q<\infty$ and $1/p+1/q=1$, is considered as the primal space. Thus, $L_p(X)$, $1<p<\infty$ and $1/p+1/q=1$, is the dual space of $L_q(X)$. In this situation, $L_q(X)$ is a separable Banach space, and therefore any weak* compact subset of $L_p(X)$ in the weak* topology is metrizable from the result of Theorem 3.16 in Rudin [9]. Consequently, any weak* compact subset of $L_p(X)$ is sequentially compact in the weak* topology.

Now, we state a proposition which is useful for this paper as follows:

Proposition 1.1. *Suppose that $f\in C(X\times X)$. If for any $k\in L_p(X)$ and every sequence $\{k_n\}$ such that $\lim_{n\to\infty}k_n=k$ in the weak* topology, then we have*:

$$\lim_{n\to\infty}\int_X\int_X f(s,t)k_n(s)\,\mathrm{d}s k_n(t)\,\mathrm{d}t=\int_X\int_X f(s,t)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t.$$

The proof of Proposition 1.1 mainly applies basic ideas of uniform continuity and uniform convergence, so we omit the proof. Here, we denote by F the feasible set of (P). By the second constraint of (P), there exists an $M>0$ such that $\|k\|_{L_p}\leqslant M$ for each $k\in F$. Hence, F is bounded in the L_p-norm. We define the set B_M as follows:

$$B_M=\{k\in L_p(X):\|k\|_{L_p}\leqslant M\}.$$

Note that the set B_M is weak* compact in the weak* topology. Then we have the following theorem.

Theorem 1.1. *Suppose that $F\neq\emptyset$. Then (P) has an optimal solution.*

Proof. Since the primal space $L_q(X)$ is a separable Banach space, B_M is metrizable. Let k be in the weak* closure of F. Note that $F\subset B_M$. There exists a sequence $\{k_i\}\subset F$ such that $\lim_{i\to\infty}k_i=k$ in the weak* topology. Since $\{k_i\}\subset F$, it follows that $\int_X\phi(s,y)k_i(s)\,\mathrm{d}s\geqslant g(y)$ for each $y\in Y$. Here, we consider $\phi\in C(X\times Y)$. Hence, $\phi(s,y)\in C(X)$ for each fixed $y\in Y$. Consequently, $\phi(s,y)\in L_q(X)$ for each fixed $y\in Y$. Applying $\lim_{i\to\infty}k_i=k$ in the weak* topology, we have $\lim_{i\to\infty}\int_X\phi(s,y)k_i(s)\,\mathrm{d}s=\int_X\phi(s,y)k(s)\,\mathrm{d}s$ for each $y\in Y$. Hence, the inequalities $\int_X\phi(s,y)k(s)\,\mathrm{d}s\geqslant g(y)$ for each $y\in Y$ follow.

Now we want to prove that $M_1\leqslant k(s)\leqslant M_2$ a.e. on X and we will do so by contradiction. There are two cases which may occur.

Case 1: There would exist a measurable subset $A\subset X$ of Lebesgue measure greater than 0, such that $k(s)<M_1$ for each $s\in A$.

Case 2: There would exist a measurable subset $B\subset X$ of Lebesgue measure greater than 0, such that $k(s)>M_2$ for each $s\in B$.

First we deal with Case 1. We denote the Lebesgue measure of A by $L(A)$. Then the characteristic function χ_A is in $L_q(X)$. Thus, we have

$$\int_X\chi_A(s)k(s)\,\mathrm{d}s=\lim_{i\to\infty}\int_X\chi_A(s)k_i(s)\,\mathrm{d}s. \tag{1}$$

This implies that

$$\int_A k(s)\,\mathrm{d}s=\lim_{i\to\infty}\int_A k_i(s)\,\mathrm{d}s\geqslant\lim_{i\to\infty}\int_A M_1\,\mathrm{d}s=M_1L(A). \tag{2}$$

From the definition of A, it follows that

$$\int_A k(s)\,\mathrm{d}s < \int_A M_1\,\mathrm{d}s = M_1 L(A). \tag{3}$$

In this situation, (2) and (3) are contradictory results. Thus, Case 1 is not possible.

As for Case 2, applying the similar technique as in Case 1 shows that Case 2 is also not possible. Consequently, we obtain the desired results $M_1 \leqslant k(s) \leqslant M_2$ a.e. on X.

According to the above results, it follows that $k \in F$, which tells us that F is weak* closed. Now, applying the Banach–Alaoglu theorem, we obtain that F is weak* compact. Let

$$V(k) = \frac{1}{2}\int_X \int_X f(s,t)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t + \int_X h(s)k(s)\,\mathrm{d}s \tag{4}$$

for any $k \in L_p(X)$. Since the weak* compact set F is metrizable in the weak* topology, by the Proposition 1.1, V is a continuous function on the weak* compact set F. Therefore, (P) has an optimal solution. □

From Theorem 1.1, we know under some condition that there exists an optimal solution for (P). In Section 2, we introduce the cutting plane method to develop an algorithm for solving (P) by a sequence of subproblems (P_n) $n=1,2,\ldots$ and prove the convergence result of this algorithm. In Section 3, we introduce the method of discretization to develop an algorithm for solving (P_n) and prove the convergence result of this algorithm. We consider the approximation solution for program (P) in Section 4. In Section 5, we give two examples to implement the proposed algorithms and see how the proposed algorithms work to solve (P_n) and (P).

2. A cutting plane algorithm and its convergence

We introduce the cutting plane method for solving (P). Let $T_n = \{y_1, y_2, \ldots, y_n\} \subset Y$. We formulate the problem (P_n) as follows:

$$\begin{aligned}
\min_{k\in L_p(X)} \quad & \frac{1}{2}\int_X \int_X f(s,t)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t + \int_X h(s)k(s)\,\mathrm{d}s \\
\text{s.t.} \quad & \int_X \phi(s,y_j)k(s)\,\mathrm{d}s \geqslant g(y_j) \quad \text{for } j=1,2,\ldots,n, \\
& 0 \leqslant M_1 \leqslant k(s) \leqslant M_2 \text{ a.e.} \quad \text{on } X.
\end{aligned}$$

Let F_n denote the feasible set of (P_n). Then we have the following theorem.

Theorem 2.1. *Suppose that $F_n \neq \emptyset$ for each $n \in N$. Then (P_n) has an optimal solution.*

Proof. The proof is similar to the proof of Theorem 1.1. □

Note that $F \subset F_n$. In the following, we denote by $V(P)$ and $V(P_n)$ the optimal values for (P) and (P_n), respectively. It is known that $V(P) \geqslant V(P_n)$. Now we propose the following algorithm by using a sequence of problems (P_n) to solve problem (P).

Algorithm 1.

Step 1: Set $n=1$, choose any $y_1 \in Y$, and set $T_1 = \{y_1\}$.

Step 2: Solve (P_n) with an optimal solution $k_n^\star$.

Step 3: Find a minimizer y_{n+1} of $\Psi_n(y)$ over Y where

$$\Psi_n(y) = \int_X \phi(s,y)k_n^\star(s)\,\mathrm{d}s - g(y) \quad \text{for each } y \in Y. \tag{5}$$

Step 4: If $\Psi_n(y_{n+1}) \geqslant 0$, then stop. In this case, $k_n^\star$ is optimal for (P). Otherwise, set $T_{n+1} = T_n \bigcup \{y_{n+1}\}$, increment $n \leftarrow n+1$, and go to Step 2.

We will show the following convergence result for Algorithm 1.

Theorem 2.2. *Suppose that $F \neq \emptyset$ and $F_n \neq \emptyset$ for each $n \in N$. Then we have* $\lim_{n\to\infty} V(P_n) = V(P)$.

Proof. By the construction of (P_n) from the Algorithm 1, we have $F_1 \supset F_2 \supset \cdots \supset F$. Consequently, $V(P_1) \leqslant V(P_2) \leqslant \cdots \leqslant V(P)$. There are three cases that may occur.

Case 1: The process would stop after a finite number of iterations.

Case 2: $\lim_{n\to\infty} V(P_n) = V(P)$.

Case 3: $\lim_{n\to\infty} V(P_n) = V(P) - \eta$, where $\eta > 0$.

When Case 1 or Case 2 occurs, then we obtain an optimal value for (P). Now, we want to show that Case 3 is not possible. Note that $\|k_n^\star\|_{L_p} \leqslant M$. It follows that $\{k_n^\star\} \subset B_M = \{k \in L_p(X) : \|k\|_{L_p} \leqslant M\}$ which is a weak* compact subset of $L_p(X)$ in the weak* topology. Since, $L_q(X)$, $1 < q < \infty$, is separable, there exists a subsequence $\{k_{n_j}^\star\}$ of $\{k_n^\star\}$, such that $k_{n_j}^\star$ is weak* convergent to some $k^\star in L_p(X)$. Applying Proposition 1.1, we have

$$\begin{aligned}
&\frac{1}{2}\int_X \int_X f(s,t)k^\star(s)\,\mathrm{d}s k^\star(t)\,\mathrm{d}t + \int_X h(s)k^\star(s)\,\mathrm{d}s \\
&\quad = \lim_{j\to\infty}\left(\frac{1}{2}\int_X\int_X f(s,t)k_{n_j}^\star(s)\,\mathrm{d}s k_{n_j}^\star(t)\,\mathrm{d}t + \int_X h(s)k_{n_j}^\star(s)\,\mathrm{d}s\right) \\
&\quad = \lim_{j\to\infty} V(P_{n_j}) \\
&\quad = V(P) - \eta.
\end{aligned}$$

Therefore, $k^\star$ does not belong to the feasible set F of (P). We define $\Psi(y)$ over Y as follows:

$$\Psi(y) = \int_X \phi(s,y)k^\star(s)\,\mathrm{d}s - g(y) \quad \text{for each } y \in Y. \tag{6}$$

Let $y^\star$ be a minimizer of $\Psi(y)$. There are three cases may occur.

Case a: $\Psi(y^\star) < 0$.

Case b: There would exist some measurable subset $A \subset X$ of Lebesgue measure greater than 0, such that $k(s) < M_1$ for each $s \in A$.

Case c: There would exist some measurable subset $B \subset X$ of Lebesgue measure greater than 0, such that $k(s) > M_2$ for each $s \in B$.

First, we deal with Case a. From the definition of $k^\star$, we have $\Psi(y_j) \geqslant 0$ for $j = 1, 2, \ldots$. Let $\{k_{j_i}^\star\}$ be a subsequence of $\{k_{n_j}^\star\}$, such that y_{j_i+1} tends toward to a limit point y'. Due to the choice of y_{j_i+1} in Algorithm 1, we find that, for each i, $\Psi_{j_i}(y^\star) \geqslant \Psi_{j_i}(y_{j_i+1})$. Applying the continuity of ϕ and g and letting $i \to \infty$, we have $\Psi(y^\star) \geqslant \Psi(y')$. Applying that $\Psi(y_j) \geqslant 0$ for $j = 1, 2, \ldots$, we have $\Psi(y^\star) \geqslant \Psi(y') \geqslant 0$. This contradicts the assumption that $\Psi(y^\star) < 0$, and hence Case a is not possible.

Applying the similar technique in Theorem 1.1, it is known that Case b and Case c are not possible. Hence, Case 3 is likewise not possible. Therefore, we complete the proof. □

3. A discretization algorithm and its convergence

We introduce the method of discretization for solving (P_n). For each $l \in N$, we define a partition Λ_l of X which satisfy

(1) $\Lambda_l = \{s_0, s_1, \ldots, s_{2^l}\}$ and
(2) $s_1 - s_0 = s_2 - s_1 = \cdots = s_{2^l} - s_{2^l-1}$.

Let $|X|$ denote the length of X. Then the partition norm of X is $|X|/2^l$. We formulate the problem $(P_{n,l})$ as follows:

$$\begin{aligned}
\min_{k\in L_p(X)} \quad & \frac{1}{2}\int_X\int_X f(s,t)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t+\int_X h(s)k(s)\,\mathrm{d}s\\
\text{s.t.} \quad & \int_X \phi(s,y_j)k(s)\,\mathrm{d}s \geqslant g(y_j) \quad \text{for } j=1,2,\ldots,n,\\
& 0\leqslant M_1\leqslant k(s)\leqslant M_2 \text{ a.e.} \quad \text{on } X,\\
& k \text{ is a step function a.e. with respect to } \Lambda_l.
\end{aligned}$$

Let $F_{n,l}$ denote the feasible set of $(P_{n,l})$ and let $V(P_{n,l})$ denote the optimal value of $(P_{n,l})$. Note that $F_{n,l}\subset B_M$.

Theorem 3.1. *Suppose that $F_{n,l}\neq\emptyset$ for each $l\in N$. Then $(P_{n,l})$ has an optimal solution.*

Proof. Let k be in the weak* closure of $F_{n,l}\subset B_M$. Then there exists a sequence $\{k_m\}\subset F_{n,l}$ such that $\lim_{m\to\infty}k_m=k$ in the weak* topology. Applying the technique as in Theorem 1.1, we obtain that:

$$\int_X \phi(s,y_j)k(s)\,\mathrm{d}s\geqslant g(y_j) \quad \text{for } j=1,2,\ldots,n \tag{7}$$

and

$$0\leqslant M_1\leqslant k(s)\leqslant M_2 \text{ a.e.} \quad \text{on } X. \tag{8}$$

Now we want to show that k is a step function a.e. with respect to Λ_l. Note that $\{k_m\}\subset F_{n,l}$. We define k_m, for each m, as follows:

$$k_m(s)=c_i^m \text{ a.e.} \quad \text{on } s\in(s_{i-1},s_i) \text{ and } \quad i=1,2,\ldots,2^l. \tag{9}$$

For each i, we have $M_1\leqslant c_i^m\leqslant M_2$ for $m=1,2,\ldots$, that is, $\{c_i^m: m=1,2,\ldots\}$ is a bounded sequence of R. Applying the elementary properties in advanced calculus, there exist a subsequence $\{m_j\}$ of $\{m\}$ and $c_i\in R$ for $i=1,2,\ldots,2^l$ such that $\lim_{j\to\infty}c_i^{m_j}=c_i$ for $i=1,2,\ldots,2^l$. Define the functions k' as follows:

$$k'(s)=c_i \quad \text{if } s\in(s_{i-1},s_i) \text{ and } i=1,2,\ldots,2^l. \tag{10}$$

Then we have $\lim_{j\to\infty}k_{m_j}(s)=k'(s)$ a.e. on X. Since $\|k_{m_j}\|_{L_p}\leqslant M$ for each j and $1<p<\infty$, applying a basic property in Royden [8], we have

$$\lim_{j\to\infty}\int_X k_{m_j}(s)\theta(s)\,\mathrm{d}s=\int_X k'(s)\theta(s)\,\mathrm{d}s \tag{11}$$

for each $\theta\in L_q(X)$ where $1/p+1/q=1$. Thus, we have $\lim_{j\to\infty}k_{m_j}=k'$ in the weak* topology.

Combining $\lim_{j\to\infty}k_{m_j}=k'$ in the weak* topology and $\lim_{j\to\infty}k_{m_j}=k$ in the weak* topology, it follows that $k=k'$ a.e. on X. If $k=k'$ a.e. on X is not true, then there are two cases that may occur.

Case 1: There would exist a measurable subset $A\subset X$ of Lebesgue measure greater than 0, such that $k(s)>k'(s)$ for each $s\in A$.

Case 2: There would exist a measurable subset $B\subset X$ of Lebesgue measure greater than 0, such that $k(s)<k'(s)$ for each $s\in B$.

First, we deal with Case 1. By the weak* convergence, it follows that:

$$\lim_{j\to\infty}\int_X k_{m_j}(s)\chi_A(s)\,\mathrm{d}s=\int_X k(s)\chi_A(s)\,\mathrm{d}s=\int_X k'(s)\chi_A(s)\,\mathrm{d}s.$$

Clearly, we have $\int_A k(s)\,\mathrm{d}s=\int_A k'(s)\,\mathrm{d}s$. However, Case 1 implies that $\int_A k(s)\,\mathrm{d}s>\int_A k'(s)\,\mathrm{d}s$. Thus, we have contradictory results and Case 1 becomes impossible. Similarly, Case 2 is also not possible. Consequently, $k=k'$ a.e. on X. Since k' is a step function with respect to Λ_l, we obtain that $k\in F_{n,l}$. This implies that $F_{n,l}$ is weak* closed. Note that $F_{n,l}$ is bounded in the L_p-norm. By the Banach–Alaoglu theorem, $F_{n,l}$ is weak* compact. As in Theorem 1.1, V is a continuous function on the weak* compact set $F_{n,l}$. Therefore, $(P_{n,l})$ has an optimal solution. □

Since $F_{n,1} \subset F_{n,2} \subset \cdots \subset F_n$, it follows that $V(P_{n,1}) \geqslant V(P_{n,2}) \geqslant \cdots \geqslant V(P_n)$. Before proving the convergence result $\lim_{l\to\infty} V(P_{n,l}) = V(P_n)$, we need some results from Royden [8]. We state them as follows:

Let $\Delta = \{\mu_0, \mu_1, \ldots, \mu_n\}$ be a partition of $[a, b]$. We define the step function φ_Δ by taking φ_Δ to be constant on each subinterval $[\mu_{k-1}, \mu_k)$ of the partition and equating to the average of f over that subinterval. We will arrive at $\|f - \varphi_\Delta\|_{L_p} \to 0$ as the length δ of the largest subinterval of Δ becomes zero.

Definition 3.1. Let $\Delta = \{\mu_0, \mu_1, \ldots, \mu_n\}$ be a partition of the finite interval $[a, b]$ and f is an integrable function on $[a, b]$. Then the function φ_Δ on $[a, b]$ defined by

$$\varphi_\Delta(x) = \frac{1}{\mu_k - \mu_{k-1}} \int_{\mu_{k-1}}^{\mu_k} f(t)\,\mathrm{d}t, \quad x \in [\mu_{k-1}, \mu_k) \tag{12}$$

is called the Δ-approximation to f in mean.

We have the following theorems from Proposition 9 and Problem 17 in Section 6.4 of Royden [8].

Theorem 3.2. *Suppose that $f \in L_p[a, b]$ and $1 \leqslant p < \infty$. Then we have $\|f - \varphi_\Delta\|_{L_p} \to 0$ as $\delta \to 0$ where δ is the length of the largest subinterval of Δ.*

Theorem 3.3. *Let $\{f_n\}$ be a sequence of functions in $L_p(X)$, $1 < p < \infty$, which converge almost everywhere to a function f in $L_p(X)$, and suppose that there is a constant M, such that $\|f_n\|_{L_p} \leqslant M$ for all n. Then for each function g in $L_q(X)$, we have*

$$\lim_{n\to\infty} \int_X f_n(s)g(s)\,\mathrm{d}s = \int_X f(s)g(s)\,\mathrm{d}s$$

Assumption 1. There exists a feasible solution $k' \in F$ which satisfies

$$\int_X \phi(s, y)k'(s)\,\mathrm{d}s > g(y) \quad \text{for each } y \in Y. \tag{13}$$

In the following, we denote $\frac{1}{2}\int_X \int_X f(s,t)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t + \int_X h(s)k(s)\,\mathrm{d}s$ by $V(k)$ for each $k \in L_p(x)$. Then we have the following theorem.

Theorem 3.4. *Suppose that $k_n \in F_n$. Let the Assumption 1 be satisfied. Then there exists a sequence $\{\varphi_{l(i)}\}$ whose each term $\varphi_{l(i)} \in F_{n,l(i)}$ converges to k_n in the* weak* *topology.*

Proof. Applying the technique stated in Royden [8] to k_n, we define the step function φ_l with respect to Δ_l as follows:

$$\varphi_l(s) = \frac{1}{s_i - s_{i-1}} \int_{s_{i-1}}^{s_i} k_n(t)\,\mathrm{d}t, \quad s \in [s_{i-1}, s_i). \tag{14}$$

By Theorem 3.2, it follows that $\lim_{l\to\infty} \|\varphi_l - k_n\|_{L_p} = 0$, and hence $\lim_{l\to\infty} \varphi_l = k_n$ in the weak* topology.

For this function k_n, there are two cases that may occur.

Case 1: $\int_X \phi(s, y_j)k_n(s)\,\mathrm{d}s > g(y_j)$ for $j = 1, 2, \ldots, n$.

Case 2: There would exist some $j \in \{1, 2, \ldots, n\}$, such that $\int_X \phi(s, y_j)k_n(s)\,\mathrm{d}s = g(y_j)$ and $\int_X \phi(s, y_j)\,k_n(s)\,\mathrm{d}s \geqslant g(y_j)$ for $j = 1, 2, \ldots, n$.

First we deal with Case 1. Applying that $\lim_{l\to\infty} \varphi_l = k_n$ in the weak* topology, there exists an $l' \in N$ such that $\int_X \phi(s, y_j)\varphi_l(s)\,\mathrm{d}s \geqslant g(y_j)$ for $j = 1, 2, \ldots, n$ and all $l \geqslant l'$. Since $0 \leqslant M_1 \leqslant k_n(s) \leqslant M_2$ a.e. on X, it is clear that $0 \leqslant M_1 \leqslant \varphi_l(s) \leqslant M_2$ a.e. on X for all $l \in N$. Hence, for this case, we obtain that $\varphi_l \in F_{n,l}$ for all $l \geqslant l'$. Thus, we have the desired result.

As for Case 2, from Assumption 1, we know that $k' \in F_n$ for each $n \in N$ and that F_n is known as a convex set. Then, for $0 < \alpha < 1$, $\alpha k' + (1-\alpha)k_n$ is in F_n. Here, we choose a decreasing sequence $\{\alpha_i\} \subset (0, 1)$ such that $\{\alpha_i\}$ converges to 0. This implies that $k_n(s) = \lim_{i\to\infty}(\alpha_i k'(s) + (1-\alpha_i)k_n(s))$ a.e. on X. Since $\|\alpha_i k' + (1-\alpha_i)k_n\|_{L_p} \leqslant M$ for each i and $1 < p < \infty$, applying Theorem 3.3, we have

$$\lim_{i\to\infty}\int_X \theta(s)(\alpha_i k'(s) + (1-\alpha_i)k_n(s))\,\mathrm{d}s = \int_X \theta(s)k_n(s)\,\mathrm{d}s \tag{15}$$

for each $\theta \in L_q(X)$ where $1/p + 1/q = 1$. Thus, we have $\lim_{i\to\infty}(\alpha_i k' + (1-\alpha_i)k_n = k_n$ in the weak* topology. It is obvious that $\int_X \phi(s, y_j)(\alpha_i k'(s) + (1-\alpha_i)k_n(s))\,\mathrm{d}s$ lies between $\int_X \phi(s, y_j)k'(s)\,\mathrm{d}s$ and $\int_X \phi(s, y_j)k_n(s)\,\mathrm{d}s$ for $j = 1, 2, \ldots, n$. Due to Assumption 1, we have

$$\int_X \phi(s, y_j)(\alpha_i k'(s) + (1-\alpha_i)k_n(s))\,\mathrm{d}s > g(y_j) \quad \text{for } j = 1, 2, \ldots, n. \tag{16}$$

For each defined l, we define the step function φ_l with respect to Λ_l as follows:

$$\varphi_l(s) = \frac{1}{s_m - s_{m-1}}\int_{s_{m-1}}^{s_m} (\alpha_i k'(t) + (1-\alpha_i)k_n(t))\,\mathrm{d}t,$$

$$s \in [s_{m-1}, s_m) \quad \text{and} \quad m = 1, 2, \ldots, 2^l.$$

By Theorem 3.2, it follows that $\lim_{l\to\infty}\|\alpha_i k' + (1-\alpha_i)k_n - \varphi_l\|_{L_p} = 0$, and hence $\lim_{l\to\infty}\varphi_l = \alpha_i k' + (1-\alpha_i)k_n$ in the weak* topology. Since $\int_X \phi(s, y_j)(\alpha_i k'(s) + (1-\alpha_i)k_n(s))\,\mathrm{d}s > g(y_j)$ for $j = 1, 2, \ldots, n$, there exists an $l(i) \in N$ such that

$$\|\alpha_i k' + (1-\alpha_i)k_n - \varphi_{l(i)}\|_{L_p} < \frac{1}{i} \tag{17}$$

and

$$\int_X \phi(s, y_j)\varphi_{l(i)}(s)\,\mathrm{d}s \geqslant g(y_j) \tag{18}$$

for $j = 1, 2, \ldots, n$. Since the definition of $\{\varphi_l\}$, we get that $M_1 \leqslant \varphi_l(s) \leqslant M_2$ a.e. on X. Thus, $\varphi_{l(i)} \in F_{n,l(i)}$.

Now, we claim that $\lim_{i\to\infty}\varphi_{l(i)} = k_n$ in the weak* topology. For each fixed $\theta \in L_q(X)$, we have:

$$\begin{aligned}
&\left|\int_X \theta(s)(\varphi_{l(i)}(s) - k_n(s))\,\mathrm{d}s\right| \\
&\quad\leqslant \left|\int_X \theta(s)(\varphi_{l(i)}(s) - (\alpha_i k'(s) + (1-\alpha_i)k_n(s)))\,\mathrm{d}s\right| \\
&\qquad + \left|\int_X \theta(s)(\alpha_i k'(s) + (1-\alpha_i)k_n(s) - k_n(s))\,\mathrm{d}s\right| \\
&\quad\leqslant \|\theta\|_{L_q}\|\alpha_i k' + (1-\alpha_i)k_n - \varphi_{l(i)}\|_{L_p} \\
&\qquad + \left|\int_X \theta(s)(\alpha_i k'(s) + (1-\alpha_i)k_n(s) - k_n(s))\,\mathrm{d}s\right| \\
&\quad\leqslant \|\theta\|_{L_q}\left(\frac{1}{i}\right) + \left|\int_X \theta(s)(\alpha_i k'(s) + (1-\alpha_i)k_n(s) - k_n(s))\,\mathrm{d}s\right| \\
&\qquad \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Hence, we get the desired result and complete the proof. □

Now, we want to show the convergent result for $\lim_{l\to\infty} V(P_{n,l}) = V(P_n)$. From Theorem 3.4, we know that for each $k_n \in F_n$ there exists a sequence $\{\varphi_{l(i)}\}$ each term $\varphi_{l(i)} \in F_{n,l(i)}$ of which converges to k_n in the weak* topology. Applying Theorem 3.4, we prove the convergence result for $\lim_{l\to\infty} V(P_{n,l}) = V(P_n)$ in the following theorem.

Theorem 3.5. *Suppose that $F_{n,l} \neq \emptyset$ for each $l \in N$ and $F_n \neq \emptyset$. Let Assumption* 1 *be satisfied, then we have $\lim_{l\to\infty} V(P_{n,l}) = V(P_n)$.*

Proof. Let $k^\star_{n,l}$ be an optimal solution of $(P_{n,l})$. We note that $\{k^\star_{n,l} : l = 1, 2, \ldots\} \subset B_M = \{k \in L_p(X) : \|k\|_{L_p} \leqslant M\}$. Applying the technique in Theorem 2.2, there exists a subsequence $\{k^\star_{n,l_i}\}$ of $\{k^\star_{n,l}\}$, such that $k^\star_{n,l_i}$ is weak* convergent to some $k'_n \in L_p(X)$. Applying the technique in Theorem 1.1, we obtain that $k'_n \in F_n$. Consequently, it follows that $\lim_{l\to\infty} V(P_{n,l}) = \lim_{i\to\infty} V(P_{n,l_i}) = \lim_{i\to\infty} V(k^\star_{n,l_i}) = V(k'_n) \geqslant V(P_n)$.We assume to the contrary that $V(k'_n) > V(P_n)$. Then there exists a $\tilde{k_n} \in F_n$, such that $V(k'_n) > V(\tilde{k_n}) > V(P_n)$. Applying Theorem 3.4 to $\tilde{k_n}$, there exists a sequence $\{\varphi_{l(i)}\}$ whose each term $\varphi_{l(i)} \in F_{n,l(i)}$ converges to $\tilde{k_n}$ in the weak* topology, and hence $\lim_{i\to\infty} V(\varphi_{l(i)}) = V(\tilde{k_n})$. Thus, there exist an $l(i) \in N$ and a $\varphi_{l(i)} \in F_{n,l(i)}$ such that:

$$V(k'_n) > V(\varphi_{l(i)}) > V(P_n). \tag{19}$$

We note that $\{V(P_{n,l}) : l = 1, 2, \ldots\}$ is decreasing, and that $V(P_{n,l}) \geqslant V(k'_n)$ for $l = 1, 2, \ldots$. Thus we have that

$$V(\varphi_{l(i)}) \geqslant V(P_{n,l(i)}) \geqslant V(k'_n). \tag{20}$$

Therefore, (19) and (20) lead to a contradiction. Hence, we obtain that $\lim_{l\to\infty} V(P_{n,l}) = V(P_n)$ and we complete the proof. □

4. Approximation solution for program (P)

It is known that $k^\star_n$ is an optimal solution of (P_n) in Algorithm 1, and $k^\star_n$ can be viewed as an approximate optimal solution of (P) from Theorem 2.2. It is important to see how good such an approximate optimal solution is. In order to attain our purpose, we need to define $\delta(k^\star_n)$, and we define $\delta(k^\star_n)$ as follows:

$$\delta(k^\star_n) = \min_{y\in Y} \int_X \phi(s, y) k^\star_n(s)\, \mathrm{d}s - g(y). \tag{21}$$

When $\delta(k^\star_n) \geqslant 0$, then it follows that $V(P) = V(P_n)$. Now we deal with the case $\delta(k^\star_n) < 0$ in the following. Firstly, we give the following assumption.

Assumption 2. There exists a $k' \in L_p(X)$ satisfying

(1) $\int_X \phi(s, y) k'(s)\, \mathrm{d}s > 1$ for each $y \in Y$ and
(2) $M_1 < \lambda_1 \leqslant k^\star_n(s) - \delta(k^\star_n) k'(s) \leqslant \lambda_2 < M_2$ a.e. on X where λ_1 and λ_2 are constants.

We estimate the error bound between $V(P)$ and $V(P_n)$. Then we have the following theorem.

Theorem 4.1. *Suppose that $\delta(k^\star_n) < 0$. Let Assumption* 2 *be satisfied. Then we have*:

$$|V(P) - V(P_n)| \leqslant |\delta(k^\star_n)| \left| \frac{1}{2} \int_X \int_X f(s, t) k^\star_n(s)\, \mathrm{d}s k'(t)\, \mathrm{d}t + \frac{1}{2} \int_X \int_X f(s, t) k'(s)\, \mathrm{d}s k^\star_n(t)\, \mathrm{d}t \right.$$
$$\left. - \frac{\delta(k^\star_n)}{2} \int_X \int_X f(s, t) k'(s)\, \mathrm{d}s k'(t)\, \mathrm{d}t + \int_X h(s) k'(s)\, \mathrm{d}s \right|.$$

Proof. From Royden [8], there exists a sequence of step functions φ_l defined from $k^\star_n$ such that $\lim_{l\to\infty} \|\varphi_l - k^\star_n\|_{L_p} = 0$, and hence $\lim_{l\to\infty} \varphi_l = k^\star_n$ in the weak* topology. Define $\bar{k_n} = k^\star_n - \delta(k^\star_n) k'$ and $\bar{\varphi}_l = \varphi_l - \delta(k^\star_n) k'$. Using (1) of

Assumption 2, it follows that:

$$\begin{aligned}
&\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s - g(y)\\
&\quad= \int_X \phi(s,y)(k_n^\star(s) - \delta(k_n^\star)k'(s))\,\mathrm{d}s - g(y)\\
&\quad= \int_X \phi(s,y)k_n^\star(s)\,\mathrm{d}s - g(y) - \delta(k_n^\star)\int_X \phi(s,y)k'(s)\,\mathrm{d}s\\
&\quad> \delta(k_n^\star) - \delta(k_n^\star) = 0 \quad \text{for each } y \in Y.
\end{aligned}$$

Thus, $\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s > g(y)$ for each $y \in Y$. Combining (2) of Assumption 2, it is clear that $\bar{k_n} \in F$. Note that $\lim_{l\to\infty} \varphi_l = k_n^\star$ in the weak* topology. It is clear that $\lim_{l\to\infty} \bar{\varphi}_l = \bar{k_n}$ in the weak* topology.

Now, we want to show that there exists an $l_1 \in N$, such that for all $l \geqslant l_1$ we have $\int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s \geqslant g(y)$ for each $y \in Y$. First, we want to claim that $\lim_{l\to\infty} \int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s = \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s$ uniformly on Y. Note that $\phi \in C(X \times Y)$. By the uniform continuity of ϕ, for each given $\varepsilon > 0$, there exists a $\delta > 0$, such that $|f(s_1,y_1) - f(s_2,y_2)| < \varepsilon$ whenever $|(s_1,y_1) - (s_2,y_2)| < \delta$. For the above $\delta > 0$ and the compactness of Y, applying the result of Lemma 12 in Section 9.3 of Royden [8], there exist $y_1, y_2, \ldots$, and $y_m \in Y$, such that for each $y \in Y$ we can find a $y_i \in \{y_i\}_{i=1}^m$ with $|y - y_i| < \delta$. Applying $\lim_{l\to\infty} \bar{\varphi}_l = \bar{k_n}$ in the weak* topology, it follows that $\lim_{l\to\infty} \int_X \phi(s,y_i)\bar{\varphi}_l(s)\,\mathrm{d}s = \int_X \phi(s,y_i)\bar{k_n}(s)\,\mathrm{d}s$ for $i = 1,2,\ldots,m$. For the above $\varepsilon > 0$, there exists an $l_0 \in N$, such that $|\int_X \phi(s,y_i)\bar{\varphi}_l(s)\,\mathrm{d}s - \int_X \phi(s,y_i)\bar{k_n}(s)\,\mathrm{d}s| < \varepsilon$ for all $l \geqslant l_0$ and $i = 1,2,\ldots,m$. Applying the fact that a weak* convergent sequence is norm bounded, there exists an $M > 0$, such that $\|\bar{\varphi}_l\|_{L_p} \leqslant M$ for each $l \in N$. Then, for each $y \in Y$ and for all $l \geqslant l_0$, we have

$$\begin{aligned}
&\left|\int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s - \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s\right|\\
&\quad\leqslant \left|\int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s - \int_X \phi(s,y_i)\bar{\varphi}_l(s)\,\mathrm{d}s\right|\\
&\qquad+ \left|\int_X \phi(s,y_i)\bar{\varphi}_l(s)\,\mathrm{d}s - \int_X \phi(s,y_i)\bar{k_n}(s)\,\mathrm{d}s\right|\\
&\qquad+ \left|\int_X \phi(s,y_i)\bar{k_n}(s)\,\mathrm{d}s - \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s\right|\\
&\quad\leqslant \varepsilon M|X|^{1/q} + \varepsilon + \varepsilon\|\bar{k_n}\|_{L_p}|X|^{1/q}.
\end{aligned}$$

Hence, $\lim_{l\to\infty} \int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s = \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s$ uniformly on Y, and it is clear that $\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s - g(y)$ is a continuous function on Y. Applying the fact that $\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s > g(y)$ for each $y \in Y$, it is obvious that $\min_{y\in Y}(\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s - g(y)) > 0$. We denote $\min_{y\in Y}(\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s - g(y))$ by τ. If we set $\varepsilon = \tau/2$, there exists an $l_1 \in N$, such that for all $l \geqslant l_1$ we have $|\int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s - \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s| < \varepsilon$ for each $y \in Y$. Then, for each $y \in Y$, it follows that:

$$\begin{aligned}
&-\varepsilon + \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s < \int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s,\\
&-\varepsilon + \int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s - g(y) < \int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s - g(y),\\
&-\varepsilon + \min_{y\in Y}\left(\int_X \phi(s,y)\bar{k_n}(s)\,\mathrm{d}s - g(y)\right) < \int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s - g(y)\\
&\tau/2 < \int_X \phi(s,y)\bar{\varphi}_l(s)\,\mathrm{d}s - g(y) \quad \text{for all } l \geqslant l_1.
\end{aligned}$$

Consequently, $\int_X \phi(s, y)\bar{\varphi}_l(s)\,\mathrm{d}s - g(y) > \tau/2 \geqslant 0$ for all $l \geqslant l_1$ and for each $y \in Y$. Hence, we have that $\int_X \phi(s, y)\bar{\varphi}_l(s)\,\mathrm{d}s \geqslant g(y)$ for each $y \in Y$ and for all $l \geqslant l_1$.

Now we want to claim that there exists an $l_2 \in N$, such that $0 \leqslant M_1 \leqslant \bar{\varphi}_l(s) \leqslant M_2$ a.e. on X for all $l \geqslant l_2$. Proving by contradiction, there are two cases that may occur.

Case 1: There exists a measurable subset $A \subset X$ of Lebesgue measure greater than 0, as well as a subsequence $\{l_i\}$ of $\{l\}$, such that $\bar{\varphi}_{l_i}(s) < M_1$ for each $s \in A$.

Case 2: There exists a measurable subset $B \subset X$ of Lebesgue measure greater than 0, as well as a subsequence $\{l_j\}$ of $\{l\}$, such that $\bar{\varphi}_{l_j}(s) > M_2$ for each $s \in B$.

Now we deal with Case 1. Since $\lim_{l\to\infty} \|\varphi_l - k_n^\star\|_{L_p} = 0$, it follows that $\lim_{i\to\infty} \|\varphi_{l_i} - k_n^\star\|_{L_p} = 0$. From Royden [8], there exists a subsequence $\{\varphi_{i_j}\}$ of $\{\varphi_{l_i}\}$ such that $\lim_{j\to\infty} \varphi_{i_j}(s) = k_n^\star(s)$ a.e. on X. In particular, it is true that $\lim_{j\to\infty} \varphi_{i_j}(s) = k_n^\star(s)$ a.e. on A, and hence we have that $\lim_{j\to\infty} \bar{\varphi}_{i_j}(s) = \bar{k}_n(s)$ a.e. on A. Suppose that the Lebesgue measure of A is $\mu > 0$. Given $0 < \delta < \mu$, applying Egoroff's theorem, there exists a measurable subset $S \subset A$ of Lebesgue measure less than δ, such that $\{\bar{\varphi}_{i_j}\}$ converges uniformly to $\bar{k}_n$ on $A - S$. Let $\varepsilon = \lambda_1 - M_1$ be given, there exists an $l_2 \in N$ such that $|\bar{\varphi}_{i_j}(s) - \bar{k}_n(s)| < \varepsilon$ for each $s \in A - S$ and for all $i_j \geqslant l_2$. Then it follows that $\bar{\varphi}_{i_j}(s) > \bar{k}_n(s) - \varepsilon \geqslant \lambda_1 - (\lambda_1 - M_1) = M_1$ for each $s \in A - S$ and for all $i_j \geqslant l_2$. Hence, we have

$$\bar{\varphi}_{i_j}(s) > M_1 \quad \text{for each } s \in A - S \quad \text{and for all } i_j \geqslant l_2. \tag{22}$$

Then (22) contradicts the assumption $\bar{\varphi}_{l_i}(s) < M_1$ for each $s \in A$ and for all l_i in Case 1. Hence, Case 1 is not possible. Applying the similar technique, Case 2 is likewise not possible. Thus, we obtain our desired result that there exists an $l_2 \in N$, such that $0 \leqslant M_1 \leqslant \bar{\varphi}_l(s) \leqslant M_2$ a.e. on X for all $l \geqslant l_2$.

Let $l' = \max\{l_1, l_2\}$. Then, for all $l \geqslant l'$, $\bar{\varphi}_l$ is in F. This implies that $V(P_n) \leqslant V(P) \leqslant V(\bar{\varphi}_l)$ for all $l \geqslant l'$. Thus, for all $l \geqslant l'$, it is obvious that $|V(P) - V(P_n)| \leqslant |V(\bar{\varphi}_l) - V(P_n)|$. From the definition of $\bar{\varphi}_l$, it follows that:

$$\begin{aligned}
V(\bar{\varphi}_l) &= \frac{1}{2}\int_X\int_X f(s,t)(\varphi_l(s) - \delta(k_n^\star)k'(s))\,\mathrm{d}s(\varphi_l(t) - \delta(k_n^\star)k'(t))\,\mathrm{d}t \\
&\quad + \int_X h(s)(\varphi_l(s) - \delta(k_n^\star)k'(s))\,\mathrm{d}s \\
&= V(\varphi_l) + \frac{-\delta(k_n^\star)}{2}\int_X\int_X f(s,t)\varphi_l(s)\,\mathrm{d}s k'(t)\,\mathrm{d}t \\
&\quad + \frac{-\delta(k_n^\star)}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}s\varphi_l(t)\,\mathrm{d}t + \frac{\delta^2(k_n^\star)}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t \\
&\quad - \delta(k_n^\star)\int_X h(s)k'(s)\,\mathrm{d}s.
\end{aligned}$$

Then we have

$$\begin{aligned}
|V(P) - V(P_n)| \leqslant |V(\varphi_l) - V(P_n)| + \Bigg| &\frac{-\delta(k_n^\star)}{2}\int_X\int_X f(s,t)\varphi_l(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t \\
&+ \frac{-\delta(k_n^\star)}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}s\varphi_l(t)\,\mathrm{d}t + \frac{\delta^2(k_n^\star)}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t \\
&-\delta(k_n^\star)\int_X h(s)k'(s)\,\mathrm{d}s\Bigg| \quad \text{for each } l
\end{aligned} \tag{$*$}$$

$$\begin{aligned}
(*) \to |\delta(k_n^\star)|\Bigg|&\frac{1}{2}\int_X\int_X f(s,t)k_n^\star(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t + \frac{1}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}sk_n^\star(t)\,\mathrm{d}t \\
&-\frac{\delta(k_n^\star)}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t + \int_X h(s)k'(s)\,\mathrm{d}s\Bigg| \quad \text{as } l \to \infty.
\end{aligned}$$

Thus we have

$$|V(P)-V(P_n)|\leqslant|\delta(k_n^\star)|\left|\frac{1}{2}\int_X\int_X f(s,t)k_n^\star(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t+\frac{1}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}sk_n^\star(t)\,\mathrm{d}t\right.$$
$$\left.-\frac{\delta(k_n^\star)}{2}\int_X\int_X f(s,t)k'(s)\,\mathrm{d}sk'(t)\,\mathrm{d}t+\int_X h(s)k'(s)\,\mathrm{d}s\right|. \qquad \square$$

5. Numerical examples

From Theorem 2.2, we have the result $\lim_{n\to\infty}V(P_n)=V(P)$. In order to find the optimal value $V(P)$, we must calculate the numerical value $V(P_n)$ first. As for how to solve (P_n), which is discussed in Section 3. In Section 3, we introduce the method of discretization to get a sequence of subproblems $(P_{n,l})$, $l=1,2,\ldots$, from (P_n). We have the result of Theorem 3.5 which says that $\lim_{l\to\infty}V(P_{n,l})=V(P_n)$. In this situation, we must calculate $V(P_{n,l})$ first, and then $V(P_n)$ can be obtained. For the purpose of numerical implementations, we give the following algorithm to evaluate the numerical value $V(P_n)$.

Algorithm 2.

Step 1: Let $\varepsilon>0$ be a sufficiently small number. Set $l=1$.

Step 2: Solve $(P_{n,l})$ with an optimal solution $k_{n,l}^\star$.

Step 3: If $0<V(P_{n,l-1})-V(P_{n,l})<\varepsilon$, then stop. In this case, $k_{n,l}^\star$ is an approximate optimal solution of (P_n). Otherwise, increment $l\leftarrow l+1$, and go to Step 2.

In the following Examples 5.1 and 5.2, we consider $\varepsilon=10^{-5}$ as the stop criterion for Step 3 of Algorithm 2. After calculating $V(P_n)$, we again need to give a stop criterion for Step 4 of Algorithm 1 to see whether this $V(P_n)$ is our desired approximate optimal value for (P) or not. Here, we let 10^{-5} be the stop criterion for Step 4 of Algorithm 1, that is, when $-10^{-5}<\delta(k_n^\star)$, then $V(P_n)$ is the approximate optimal value for (P). For this end, the MATLAB (version 7.0) was installed on a PC for our study. Let us consider the following examples.

Example 5.1.

$$\begin{aligned}\min_{k\in L_p[-1,1]}\quad & \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}((s-t)^2-2)^2k(s)\,\mathrm{d}sk(t)\,\mathrm{d}t+\int_{-1}^{1}k(s)\,\mathrm{d}s\\ \text{s.t.}\quad & \int_{-1}^{1}((s-y)^2-2)^2k(s)\,\mathrm{d}s\geqslant 1,\quad \text{for each } y\in Y,\\ & 0.1\leqslant k(s)\leqslant 1 \text{ a.e.}\quad \text{on } X.\end{aligned}$$

with $X=Y=[-1,1]$, $f(s,t)=((s-t)^2-2)^2$, $h(s)=1$, $\phi(s,y)=((s-y)^2-2)^2$, $g(y)=1$, $M_1=0.1$, and $M_2=1$.

We start $y_1=-1$ in Algorithm 1. Now we start Algorithm 1 with solving (P_1). We solve (P_1) by applying Algorithm 2 and $\varepsilon=10^{-5}$. Then we have the following output: $V(P_{1,8})-V(P_{1,9})<10^{-5}$. Hence, we stop Algorithm 2 and $V(P_{1,9})$ is the approximate optimal value of (P_1). Let y_2 be a minimizer of $\Psi_1(y)$ over Y as in Algorithm 1. From the output we now have the result Table 1.

Note that $k_{1,9}^\star$ is an approximate optimal solution of (P_1) from Algorithm 2. From the implementation results, we can define $k_{1,9}^\star$ as follows:

$$k_{1,9}^\star(s)=\begin{cases}0.1 & \text{if } s\in(-217/256,\,249/256),\\ 0.4 & \text{if } s\in(249/256,\,250/256),\\ 0.8 & \text{if } s\in(-218/256,\,-217/256),\\ 1 & \text{if } s\in(-1,\,-218/256)\cup(250/256,\,1).\end{cases} \tag{23}$$

Table 1

$V(P_{1,8}) - V(P_{1,9})$	$V(P_{1,9})$	$\delta(k_1^\star)$	y_2
$< 10^{-5}$	0.513395	−0.455904	0.589515

Table 2

$V(P_{2,8}) - V(P_{2,9})$	$V(P_{2,9})$	$\delta(k_2^\star)$	y_3
$< 10^{-5}$	0.586792	−0.321862	−0.585425

The results in Table 1 tell us that $\delta(k_1^\star) < -10^{-5}$. Thus, we go to Step 2 in Algorithm 1 and continue to solve (P_2). Using the same arguments as above, we have the result Table 2 and

$$k_{2,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -201/256) \cup (-186/256, 213/256), \\ 0.9 & \text{if } s \in (213/256, 214/256), \\ 1 & \text{if } s \in (-201/256, -186/256) \cup (214/256, 1). \end{cases} \tag{24}$$

From the results in Table 2, we stop Algorithm 2 and have $V(P_{2,9})$ as the approximate optimal value of (P_2). Since $\delta(k_2^\star) < -10^{-5}$, we go to Step 2 in Algorithm 1 and we continue to solve (P_3). Note that y_3 is a minimizer of $\Psi_2(y)$ over Y as in Algorithm 1. Repeating the same arguments as above, we can obtain the following iterations and results (Tables 3–8)

$$k_{3,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -95/256) \cup (-54/256, 230/256), \\ 0.5 & \text{if } s \in (-55/256, -54/256), \\ 0.6 & \text{if } s \in (-95/256, -94/256) \cup (230/256, 231/256), \\ 1 & \text{if } s \in (-94/256, -55/256) \cup (231/256, 1), \end{cases} \tag{25}$$

$$k_{4,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -132/256) \cup (-94/256, 221/256), \\ 0.2 & \text{if } s \in (-95/256, -94/256) \cup (221/256, 222/256), \\ 0.3 & \text{if } s \in (-132/256, -131/256), \\ 1 & \text{if } s \in (-131/256, -95/256) \cup (222/256, 1). \end{cases} \tag{26}$$

$$k_{5,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -116/256) \cup (-76/256, 225/256), \\ 0.5 & \text{if } s \in (-77/256, -76/256) \cup (225/256, 226/256), \\ 1 & \text{if } s \in (-116/256, -77/256) \cup (226/256, 1). \end{cases} \tag{27}$$

$$k_{6,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -116/256) \cup (-74/256, 226/256), \\ 0.5 & \text{if } s \in (-75/256, -74/256), \\ 0.6 & \text{if } s \in (-116/256, -115/256), \\ 1 & \text{if } s \in (-115/256, -75/256) \cup (226/256, 1). \end{cases} \tag{28}$$

$$k_{7,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -118/256) \cup (-76/256, 225/256), \\ 0.3 & \text{if } s \in (-77/256, -76/256) \cup (225/256, 226/256), \\ 0.6 & \text{if } s \in (-118/256, -117/256), \\ 1 & \text{if } s \in (-117/256, -77/256) \cup (226/256, 1). \end{cases} \tag{29}$$

Table 3

$V(P_{3,8}) - V(P_{3,9})$	$V(P_{3,9})$	$\delta(k_3^\star)$	y_4
$< 10^{-5}$	0.660357	−0.058272	−0.817254

Table 4

$V(P_{4,8}) - V(P_{4,9})$	$V(P_{4,9})$	$\delta(k_4^\star)$	y_5
$< 10^{-5}$	0.681130	−0.015646	−0.709855

Table 5

$V(P_{5,8}) - V(P_{5,9})$	$V(P_{5,9})$	$\delta(k_5^\star)$	y_6
$< 10^{-5}$	0.683422	−0.003723	−0.765302

Table 6

$V(P_{6,8}) - V(P_{6,9})$	$V(P_{6,9})$	$\delta(k_6^\star)$	y_7
$< 10^{-5}$	0.685164	-7.855039×10^{-5}	−0.773138

Table 7

$V(P_{7,8}) - V(P_{7,9})$	$V(P_{7,9})$	$\delta(k_7^\star)$	y_8
$< 10^{-5}$	0.685176	-2.252523×10^{-5}	−0.769230

Table 8

$V(P_{8,8}) - V(P_{8,9})$	$V(P_{8,9})$	$\delta(k_8^\star)$	y_9
$< 10^{-5}$	0.685184	-2.687748×10^{-6}	−0.768406

$$k_{8,9}^\star(s) = \begin{cases} 0.1 & \text{if } s \in (-1, -118/256) \cup (-76/256, 225/256), \\ 0.4 & \text{if } s \in (225/256, 226/256), \\ 0.8 & \text{if } s \in (-77/256, -76/256), \\ 1 & \text{if } s \in (-118/256, -77/256) \cup (226/256, 1). \end{cases} \tag{30}$$

From the implementation results, we have that $k_{3,9}^\star$, $k_{4,9}^\star, \ldots, k_{8,9}^\star$ are approximate optimal solutions of (P_3), (P_4), …, and (P_8), respectively. Moreover, $V(P_{3,9}) = 0.660357$, $V(P_{4,9}) = 0.681130, \ldots$, and $V(P_{8,9}) = 0.685184$ are the approximate optimal values of (P_3), (P_4), …, and (P_8), respectively. Here, we consider $\varepsilon = 10^{-5}$ in Algorithm 1. Using the results from Tables 3–8, we have that $-10^{-5} < \delta(k_8^\star) < 0$. Hence, we stop Algorithm 1. This tells us that $V(P_8)$ is our desired approximate optimal value of (P). From Table 8, it follows that $V(P_{8,9}) = 0.685184$ is the approximate optimal value of (P_8) according to $\varepsilon = 10^{-5}$ in Algorithm 2. Consequently, 0.685184 is the approximate optimal value of (P) according to $\varepsilon = 10^{-5}$ in Algorithm 1.

Table 9

$V(P_{1,8}) - V(P_{1,9})$	$V(P_{1,9})$	$\delta(k_1^\star)$	y_2
$< 10^{-5}$	0.335008	−0.418588	0.465570

Table 10

$V(P_{2,8}) - V(P_{2,9})$	$V(P_{2,9})$	$\delta(k_2^\star)$	y_3
$< 10^{-5}$	0.800018	−0.011412	0.546424

Table 11

$V(P_{3,8}) - V(P_{3,9})$	$V(P_{3,9})$	$\delta(k_3^\star)$	y_4
$< 10^{-5}$	0.820883	-7.948840×10^{-6}	0.548581

Example 5.2.

$$\begin{aligned} \min_{k\in L_p[0,1]} \quad & \frac{1}{2}\int_0^1\int_0^1 (s^2+t^2)k(s)\,\mathrm{d}s k(t)\,\mathrm{d}t + \int_0^1 \sin(4\pi s)k(s)\,\mathrm{d}s \\ \text{s.t.} \quad & \int_0^1 \left(\frac{s+1}{y+1}\right) k(s)\,\mathrm{d}s \geqslant -y^2+1, \quad \text{for each } y\in Y, \\ & 1\leqslant k(s)\leqslant 2 \text{ a.e.} \quad \text{on } X. \end{aligned}$$

with $X = Y = [0, 1]$, $f(s, t) = s^2 + t^2$, $h(s) = \sin(4\pi s)$, $\phi(s, y) = (s + 1)/(y + 1)$, $g(y) = -y^2 + 1$, $M_1 = 1$, and $M_2 = 2$.

We start $y_1 = 0$ in Algorithm 1. Applying the same procedure as that in Example 5.1, we have the following iterations and results (Tables 9–11).

$$k_{1,9}^\star(s) = \begin{cases} 1 & \text{if } s \in (0, 132/512) \cup (246/512, 414/512) \cup (468/512, 1), \\ 1.7 & \text{if } s \in (414/512, 415/512), \\ 2 & \text{if } s \in (132/512, 246/512) \cup (415/512, 468/512). \end{cases} \tag{31}$$

$$k_{2,9}^\star(s) = \begin{cases} 1 & \text{if } s \in (27/512, 95/512) \cup (287/512, 361/512), \\ 1.4 & \text{if } s \in (286/512, 287/512), \\ 2 & \text{if } s \in (0, 27/512) \cup (95/512, 286/512) \cup (361/512, 1). \end{cases} \tag{32}$$

$$k_{3,9}^\star(s) = \begin{cases} 1 & \text{if } s \in (29/512, 94/512) \cup (289/512, 358/512), \\ 1.9 & \text{if } s \in (358/512, 359/512), \\ 2 & \text{if } s \in (0, 29/512) \cup (94/512, 289/512) \cup (359/512, 1). \end{cases} \tag{33}$$

We note that $k_{1,9}^\star$, $k_{2,9}^\star$, and $k_{3,9}^\star$ are the approximate optimal solutions of (P_1), (P_2), and (P_3), respectively. Since $-10^{-5} < \delta(k_3^\star) < 0$, we obtain the approximate optimal value of (P) as 0.820883.

References

[1] E.J. Anderson, P. Nash, Linear Programming in Infinite-dimensional Spaces, Wiley, Chichester, New York, Brisbane and Toronto, 1987.
[2] S.C. Fang, C.J. Lin, S.Y. Wu, Solving quadratic semi-infinite programming problems by using relaxed cutting-plane scheme, J. Comput. Appl. Math. 129 (2001) 89–124.
[3] S.C. Fang, C.J. Lin, S.Y. Wu, Solving general capacity problem by relaxed cutting plane approach, Ann. Oper. Res. 103 (2001) 193–211.
[4] R. Hettich, K. Kortanek, Semi-infinite programming: theory, method, and application, SIAM Rev. 35 (1993) 380–429.
[5] S. Ito, Y. Liu, K.L. Teo, S.Y. Wu, A numerical approach to infinite-dimensional linear programming in L_1 spaces, submitted for publication.
[6] H.C. Lai, S.Y. Wu, Extremal points and optimal solutions for general capacity problems, Math. Programming 54 (1992) 87–113.
[7] D.G. Luenberger, Linear and Nonlinear Programming, Addison-Wesley, MA, 1984.
[8] H.L. Royden, Real Analysis, Prentice-Hall, New Jersey, 1988.
[9] W. Rudin, Functional Analysis, McGraw-Hill, New York, 1973.
[10] E.W. Sachs, Semi-infinite programming in control, in: R. Reemtsen, J.J. Ruckmann (Eds.), Semi-Infinite Programming, Kluwer Academic, Netherlands, 1998, pp. 389–411.
[11] R.J. Vanderbei, Extreme optics and the search for earth-like planets, Math. Programming, 2007, to appear.
[12] S.Y. Wu, A cutting plane approach to solving quadratic infinite programs on measure spaces, J. Global Optim. 21 (2001) 67–87.